Forcing NS_{ω_1} is ω_1-Dense From Large Cardinals Part III

A Journey Guided by the Stars

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Recap

Notation

Convention

Ideal means normal uniform ideal on ω_1 in this talk.

- If $\mathcal{I}$ is an ideal then $\mathbb{P}_{\mathcal{I}}$ is the associated forcing. It is

$$P(\omega_1)/\sim_{\mathcal{I}} - \{[\varnothing]_{\sim_{\mathcal{I}}}\}$$

  with the order induced by inclusion. Here, $A \sim_{\mathcal{I}} B$ iff $A \triangle B \in \mathcal{I}$.
- If G is $\mathbb{P}_{\mathcal{I}}$-generic over V then $U_G = \{A \mid [A]_{\sim_{\mathcal{I}}} \in G\}$ is a V-ultrafilter which induces the generic ultrapower

$$j_G \colon V \to \mathrm{Ult}(V, U_G).$$

Main Result

Definition

An ideal $\mathcal{I}$ is ω_1-dense if $\mathbb{P}_{\mathcal{I}}$ has a dense subsets of size ω_1.
That is there is $\langle S_i \mid i < \omega_1 \rangle$ a sequence of subsets of ω_1 so that for any $A \in \mathcal{I}^+$ there is $i < \omega_1$ with $S_i \backslash A \in \mathcal{I}$.

Theorem (L.)

If there is an inaccessible κ which is a limit of $<\kappa$-supercompact cardinals then there is a stationary set preserving forcing $\mathbb{P}$ with

$$V^{\mathbb{P}} \models \text{“}\mathrm{NS}_{\omega_1} \textit{ is } \omega_1\textit{-dense”}.$$

The Strategy

Motivated by Asperó-Schindler, $\mathrm{MM}^{++} \Rightarrow (*)$, there should be some forcing axiom FA which solves

$$\frac{\mathrm{MM}^{++}}{(*)} = \frac{\mathrm{FA}}{\mathbb{Q}_{\max}\text{-}(*)}.$$

So FA implies "NS_{ω_1} is ω_1-dense". Force FA as follows:

- Iterate small nice-ish forcings up to a supercompact κ via a RCS-iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$.
- Invoke an iteration theorem to argue that ω_1 (and suitable additional structure) is preserved along the iteration.
- Employ Baumgartner's argument to get the forcing axiom.

"NS_{ω_1} is ω_1-dense" in $V^{\mathbb{P}}$ is witnessed by a sequence $\vec{S} = \langle S_i \mid i < \omega_1 \rangle$ of stationary sets. $\mathbb{P}$ is κ-cc, so $\vec{S} \in V^{\mathbb{P}_\alpha}$ for some $\alpha < \kappa$.

- Most likely, NS_{ω_1} is not ω_1-dense in $V^{\mathbb{P}_\alpha}$.
- But then $\mathbb{P}_{\alpha,\kappa}$ **must kill stationary sets** of $V^{\mathbb{P}_\alpha}$. That's fine!
- $\mathbb{P}_{\alpha,\kappa}$ **preserves the Π_1-properties of $\vec{S}$ that hold in $V^{\mathbb{P}}$** Today!.

$\Diamond(\mathbb{B})$ and $\Diamond^+(\mathbb{B})$

More generally $\diamondsuit(\mathbb{B})$ and $\diamondsuit^+(\mathbb{B})$

Definition

Let $\mathbb{B} \subseteq \omega_1$ be a forcing. $\diamondsuit(\mathbb{B})$ holds if there is an embedding $\pi\colon \mathbb{B} \to \mathcal{P}(\omega_1)\backslash\mathrm{NS}_{\omega_1}$ so that $\forall p \in \mathbb{B}$ there are stationarily many countable $X \prec H_{\omega_2}$ with

$$p \in \{q \in \mathbb{B} \cap X \mid \omega_1 \cap X \in \pi(q)\} \text{ is a filter generic over } X.$$

We call such X π-slim.
The stronger $\diamondsuit^+(\mathbb{B})$ holds if there is π witnessing $\diamondsuit(\mathbb{B})$ so that every $X \prec H_\theta$ with $f, \mathbb{B} \in X$ is π-slim.

$\diamondsuit^+(\mathbb{B})$ is just a complete embedding $\pi\colon \mathbb{B} \to \mathcal{P}(\omega_1)\backslash\mathrm{NS}_{\omega_1}$.

Lemma (Essentially Woodin)

$\pi\colon \mathbb{B} \to \mathcal{P}(\omega_1)\backslash\mathrm{NS}_{\omega_1}$ witnesses $\diamondsuit(\mathbb{B})$ iff $[\cdot]_{\mathrm{NS}_{\omega_1}} \circ \pi\colon \mathbb{B} \to (\mathbb{P}_{\mathrm{NS}_{\omega_1}})^W$ is a complete embedding in some outer model W.

The Forcing Axiom QM

Definition

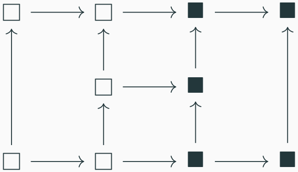
QM is the axiom: $\exists\pi$ witnessing $\diamondsuit(\omega_1^{<\omega})$ so that

$$\mathrm{FA}_{\omega_1}(\{\mathbb{P} \mid V^{\mathbb{P}} \models \text{“}\pi \text{ witnesses } \diamondsuit(\omega_1^{<\omega})\text{”}\})$$

holds.

QM implies...

- there is a Suslin tree,
- “almost disjoint coding” fails,
- the Cichon diagram is



- $\mathrm{SRP} \wedge \neg\mathrm{MRP}$.

QM is what we are looking for!

Lemma
QM *implies* NS_{ω_1} *is* ω_1*-dense!*

Proof Sketch.

- Let π witness $\diamondsuit(\omega_1^{<\omega})$. Want to show that π is a dense embedding.
- If not, let $S \in \mathrm{NS}_{\omega_1}^+$ with no set in $\mathrm{ran}(\pi)$ below S.
- Can show that $\mathrm{CS}(\omega_1 - S)$ is π-preserving.
- But by QM applied to $\mathrm{CS}(\omega_1 - S)$, $H_{\omega_2} \prec_{\Sigma_1} (H_{\omega_2})^{V^{\mathrm{CS}(\omega_1 - S)}}$, contradiction.

□

The real challenge is to force QM.

Parametrized Properness

Definition

Suppose π witnesses $\diamondsuit(\mathbb{B})$. A forcing $\mathbb{P}$ is π-**proper** if: Whenever

- $X \prec H_\theta$ countable and π-slim, $\mathbb{P} \in X$
- $p \in \mathbb{P} \cap X$

Then there is $(X, \mathbb{P}, \pi)$-generic $q \leqslant p$, i.e. forces

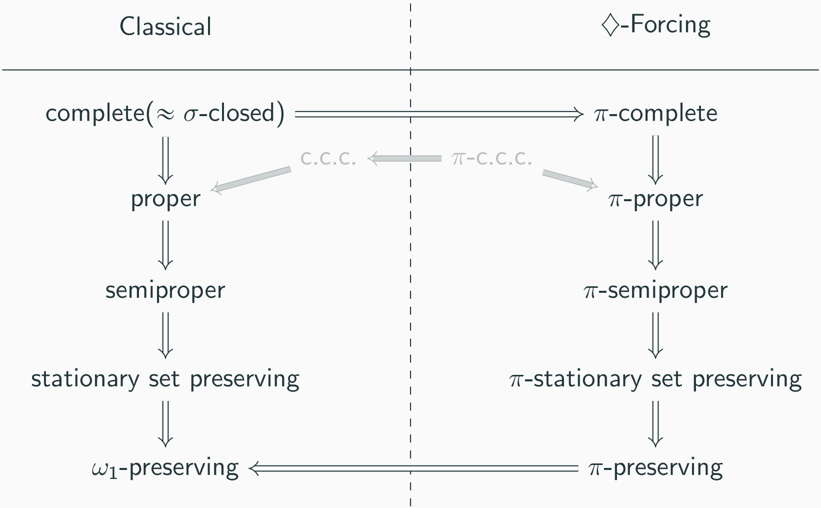
$$X = X[G] \cap V \wedge X[G] \text{ is } \pi\text{-slim.}$$

Analogously, define π-semiproperness.

Definition

Suppose π witnesses $\diamondsuit(\mathbb{B})$. A set $S \subseteq \omega_1$ is π-stationary if for large enough regular θ and all clubs $\mathcal{C} \subseteq [H_\theta]^\omega$ there is some π-slim $X \in \mathcal{C}$, $X \prec H_\theta$ with $\delta^X \in S$.

Parametrized Properness



Parametrized Properness

Some examples...

$\mathbb{B} = ...$	$\{\mathbb{1}\}$	T a Suslin tree
π-proper is...	proper	proper + T-preserving
π-semiproper is...	semiproper	semiproper + T-preserving

$\mathbb{B} = ...$	Cohen forcing
π-proper is...	"proper for a weakly Luzin sequence"
π-semiproper is...	"semiproper for a weakly Luzin sequence"

We really only care about $\mathbb{B} = \mathrm{Col}(\omega, \omega_1)$.

Iteration Theorems

Suppose π witnesses $\diamondsuit(\mathbb{B})$.

Theorem

Countable support iterations of π-proper forcings are π-proper

Theorem

RCS iterations of π-semiproper forcings are π-semiproper.

Corollary (Shelah)

Proper (semiproper) forcings are closed under countable (RCS) support iterations.

Corollary (Essentially Miyamoto)

Suppose T is a Suslin tree. Proper (semiproper) + T-preserving forcings are closed under countable (RCS) support iterations.

The Point of π-Semiproperness

We only want to iterate π-semiproper forcings here for π a witness of $\diamondsuit(\omega_1^{<\omega})$.

Corollary

If there is a supercompact cardinal then there is a π-semiproper (and hence π-preserving) poset forcing SRP.

Corollary

If there is a Woodin cardinal then there is a π-semiproper (and hence π-preserving) poset forcing "NS_{ω_1} is saturated".

Forcing QM

To force QM we need to

- force a witness π of $\diamondsuit(\omega_1^{<\omega})$ (easy)
- and then iterate arbitrary π-preserving forcings and preserve π (hard).
- Iterating π-semiproper forcings gives the forcing axiom for all π-stationary set preserving forcings, but that is not enough!

The iteration theorem from Part II generalizes.

Theorem

Suppose μ witnesses $\diamondsuit(\mathbb{B})$. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be a RCS-iteration of μ-preserving forcings and assume that for all $\alpha < \gamma$:

- $\Vdash_{\mathbb{P}_{\alpha+1}} \mathrm{SRP}$
- $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ *preserves μ-stationary sets from* $\bigcup_{\beta<\alpha} V[\dot{G}_\beta]$"

Then $\mathbb{P}$ preserves μ.

Q-Iterations

We need to get around the restriction of preserving old stationary sets. Suppose π witnesses $\diamondsuit(\omega_1^{<\omega})$.

Definition

A Q-iteration is a RCS iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ of π-preserving forcings so that for all $\alpha < \gamma$

- $\Vdash_{\mathbb{P}_\alpha+2} \mathrm{SRP}$
- $\Vdash_{\mathbb{P}_\alpha+1}$ "$\dot{\mathbb{Q}}_{\alpha+1}$ makes π dense for sets in $V[\dot{G}_{\alpha+1}]$".

Corollary (Work-Life-Balance Theorem)

Q-iteration preserve π.

This means we can force QM from large cardinals provided we find the $\dot{\mathbb{Q}}_{\alpha+1}$ which make "π dense for ground model sets" ("sealing forcings for ω_1-density").

The New Sealing Forcing

$\mathrm{MM}^{++} \Rightarrow (*)$

Assuming H_{ω_2} is a "big $\mathbb{P}_{\max}$-condition", Asperó-Schindler construct a forcing $\mathbb{P}$ so that in $V^{\mathbb{P}}$ the following picture exists:

$$
\begin{array}{ccccc}
 & & D^* & & \\
 & & \Cup & & \\
 & & q_0 & \xrightarrow{\sigma_{0,\omega_1}} & q_{\omega_1} = (N^*, I^*, b^*) \\
 & & \Cup & & \Cup \\
p_0 & \xrightarrow{\mu_{0,\omega_1^N}} & p_{\omega_1^N} & \xrightarrow{\mu_{\omega_1^N,\omega_1}} & p_{\omega_1} \\
\Cap & & & & \| \\
\mathbb{P}_{\max} & & & & ((H_{\omega_2})^V, (\mathrm{NS}_{\omega_1})^V, A)
\end{array}
$$

- μ_{0,ω_1^N} witnesses $q_0 <_{\mathbb{V}_{\max}} p_0$ and $\mu_{\omega_1^N,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^N})$.
- The top iteration $q_0 \to q_{\omega_1}$ is *correct* in $V^{\mathbb{P}}$, i.e. $I^* = (\mathrm{NS}_{\omega_1})^{V^{\mathbb{P}}} \cap N^*$.

Modifications

We want to replace $\mathbb{P}_{\max}$ by $\mathbb{Q}_{\max}$. Immediate problem: Then we have to assume that $(H_{\omega_2}, \mathrm{NS}_{\omega_1})$ is (part of) a big $\mathbb{Q}_{\max}$-condition. So NS_{ω_1} must already be ω_1-dense!

Definition
$\mathbb{Q}^-_{\max}$-conditions are of the form (M, I, π) with:

- (M, I) is generically iterable.
- $M \models$ "π witnesses $\diamondsuit^+_I(\omega_1^{<\omega})$"

$q = (N, J, \tau) <_{\mathbb{Q}^-_{\max}} (M, I, \pi) = p$ iff in N there is a generic iteration (map) $j : p \to p^* = (M^*, I^*, \pi^*)$ such that:

- $\pi^* = \tau$
- τ is dense for sets in M^*, i.e. if $S \in \mathcal{P}(\omega_1)^{M^*}$ then
  - either $S \in J$
  - or $\exists p \in \mathrm{Col}(\omega, \omega_1^N)\ \tau(p) \subseteq S \mod J$.

$\mathbb{Q}_{\max}$ embeds densely into $\mathbb{Q}^-_{\max}$ (assuming $\mathrm{AD}^{L(\mathbb{R})}$).

Does it work now?

We can force $(H_{\omega_2}, \mathrm{NS}_{\omega_1}, \pi)$ to be a "big $\mathbb{Q}^-_{\max}$-condition" using π-semiproper forcing. Following Asperó-Schindler, we get:

$$\begin{array}{ccccc}
 & & q_0 & \xrightarrow{\sigma_{0,\omega_1}} & q_{\omega_1} = (N^*, I^*, \tau^*) \\
 & & \Cup & & \Cup \\
p_0 & \xrightarrow{\mu_{0,\omega_1^N}} & p_{\omega_1^N} & \xrightarrow{\mu_{\omega_1^N,\omega_1}} & p_{\omega_1} \\
\Cap & & & & \| \\
\mathbb{Q}^-_{\max} & & & & ((H_{\omega_2})^V, (\mathrm{NS}_{\omega_1})^V, \pi)
\end{array}$$

- μ_{0,ω_1^N} witnesses $q_0 <_{\mathbb{V}_{\max}} p_0$ and $\mu_{\omega_1^N,\omega_1} = \sigma_{0,\omega_1}(\mu_{0,\omega_1^N})$.
- The top iteration $q_0 \to q_{\omega_1}$ is *correct* in $V^{\mathbb{P}}$, i.e. $I^* = (\mathrm{NS}_{\omega_1})^{V^{\mathbb{P}}} \cap N^*$.

So $\mathbb{P}$ makes π dense for sets in V, great! But this it preserve π? Unclear!!

$\diamondsuit$-Iterations

Definition

A generic iteration $\langle (M_\alpha, I_\alpha), \mu_{\alpha,\beta} \mid \alpha \leqslant \beta \leqslant \omega_1 \rangle$ is a $\diamondsuit$-iteration if: For any sequence $\langle D_i \mid i < \omega_1 \rangle$ of dense subsets of $(\mathcal{P}(\omega_1)^{M_{\omega_1}}/I_{\omega_1})^+$ and any $S \in I_{\omega_1}^+ \cap M_{\omega_1}$ have

$$\{\alpha \in S \mid \forall i < \alpha \; U_\alpha \cap \mu_{\alpha,\omega_1}^{-1}[D_i] \neq \emptyset\} \in \mathrm{NS}_{\omega_1}^+$$

where U_α is the generic ultrafilter applied to M_α.

All $\diamondsuit$-iterations are correct in the sense that if $(M^*, \mathcal{I}^*)$ is the final model of a $\diamondsuit$-iteration then $\mathcal{I}^* = \mathrm{NS}_{\omega_1} \cap M^*$. But more structure is preserved now! E.g. if $T \in M^*$ is a Suslin tree in M^* then T is really Suslin.

Final Modifications

Even better:

Lemma

Suppose $(M^*, \mathcal{I}^*)$ *is the final model of a* $\diamondsuit$*-iteration. If*

$$(M^*; \in, \mathcal{I}^*) \models \text{``}\pi \textit{ witnesses } \diamondsuit^+_{\mathcal{I}^*}(\mathbb{B})$$

then π *witnesses* $\diamondsuit(\mathbb{B})$ *in* V.

Theorem (L.)

Can modify Asperó-Schindler's $\mathbb{P}$ *to* $\mathbb{P}_\diamondsuit$ *so that in* $V^{\mathbb{P}_\diamondsuit}$ *the same picture as before exists and* $q_0 \to q_{\omega_1}$ *is a* $\diamondsuit$*-iteration in* $V^{\mathbb{P}_\diamondsuit}$.

This is the final piece! We can get our sealing forcings from Woodin cardinals!

Results

Corollary

QM *implies* $\mathbb{Q}_{\max}$-$(*)$.

Theorem

If there is a supercompact limit of supercompact cardinals then QM *holds in a stationary set preserving forcing extension.*

Theorem

If there is an inaccessible κ *which is a limit of* $<\kappa$*-supercompact cardinals then there is a stationary set preserving* $\mathbb{P}$ *with*

$$V^{\mathbb{P}} \models \text{“}\mathrm{NS}_{\omega_1} \textit{ is } \omega_1\textit{-dense”}.$$

The Mystery

The Mystery

How much can the large cardinal assumption of the main theorem be reduced? We used

- an inaccessible on the top to "catch our tail",
- Woodin cardinals for the "new sealing forcing" and
- (partial) supercompact to satisfy the greedy iteration theorem.

If we could do without SRP, we could plausibly lower the assumption to an inaccessible limit of Woodin cardinals!

Theorem (Woodin)

The large cardinal assumption of the main theorem cannot be reduced to an inaccessible limit of Woodin cardinals. In fact, consistently there is a model with an inaccessible limit of Woodin cardinals but no ω_1-preserving poset forcing "NS_{ω_1} is ω_1-dense".

At last, some Inner Model Theory!

Proof.

- Work in the least inner model M with an inaccessible limit of Woodin cardinals and a proper class of Woodin cardinals.
- Suppose $M[G] \models$ "NS_{ω_1} is ω_1-dense" and $\omega_1^M = \omega_1^{M[G]}$.
- We show that in an extension of $M[G]$, there are divergent models of AD (theorem then follows from gap in consistency strengths).
- In M, we have $\heartsuit$:

$$\forall \alpha < \omega_1 \exists x \in \mathbb{R} \ (x \text{ codes } \alpha \wedge x \in \mathrm{OD}^{L(A,\mathbb{R})} \text{ for some } A \in \mathrm{uB}) \quad (\heartsuit)$$

- Why? Let $\beta < \omega_1$ so that $M \| \beta \ni x$ some code for α. For $\Sigma = (\omega, \omega_1, \omega_1)$-iteration strategy for $M \| \beta$, have $x \in \mathrm{OD}^{L(\Sigma,\mathbb{R})}$.
- $\heartsuit$ still holds in $M[G]$!

At last, some Inner Model Theory!

Proof continued.

- Let g be $M[G]$-generic for $\mathbb{P}_{\mathrm{NS}_{\omega_1}} \cong \mathrm{Col}(\omega, \omega_1)$.
- Generic embedding $j_g \colon M[G] \to N$.
- By $\heartsuit$ in N, let x code ω_1^M, $x \in \mathrm{OD}^{L(A,\mathbb{R}^N)}$, $L(A, \mathbb{R}^N) \models \mathrm{AD}$.
- Now, $\mathbb{R}^N = \mathbb{R}^{M[G][g]}$. If there are no divergent models in $M[G][g]$ then $L(A, \mathbb{R}^N)$ is definable in $M[G][g]$ from $\Theta^{L(A,\mathbb{R}^N)}$.
- But then x is $\mathrm{OD}^{M[G][g]}$, so $x \in M[G]$ by homogeneity of $\mathrm{Col}(\omega, \omega_1)$, contradiction!

$\square$

Thank you for listening!